Connectedness of Random Walk Segmentation *

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Abstract

Connectedness of random walk segmentation is examined, and novel properties are discovered, by considering electrical circuits equivalent to random walks. A theoretical analysis shows that earlier conclusions concerning connectedness of random walk segmentation results are incorrect, and counterexamples are demonstrated.

Keywords: Image segmentation, random walk, Laplace's equation, counterexample, connectedness.


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@ARTICLE{pami11/ChengZ,
    author={Ming-Ming Cheng and Guo-Xin Zhang},
    journal={Pattern Analysis and Machine Intelligence, IEEE Transactions on},
    title={Connectedness of Random Walk Segmentation},
    year={2011},
    month=jan. ,
    volume={33},
    number={1},
    pages={200 -202},
    doi={10.1109/TPAMI.2010.138},
    ISSN={0162-8828},
}
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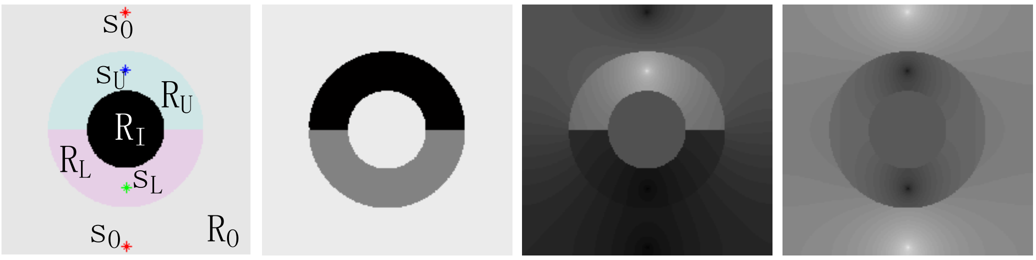
1 Introduction

The random walk algorithm proposed by Grady [1] is a leading method for seeded image segmentation. In this graph-based algorithm, edge weights denote the likelihood that a random walk will cross that edge. For each pixel (node), the probability is computed of a random walk starting at it first reaching each seed in turn. These probabilities are compared, and the pixel given the same label as the seed with greatest probability. (Multiple seeds may share the same label). An important property given by Grady's paper is that each segmented region is guaranteed to be connected to one or more seeds with that region's label: isolated regions without seeds do not occur. This is important, as it implies that the random walk approach avoids the noisy or fragmented segmentations that can sometimes result from other algorithms. Unfortunately, as we show here, this property does not always hold. We give a counterexample in Section 2 and a theoretical analysis in Section 3.

2 Counterexample and explanation

First, we give a specific counterexample, showing that the connectedness property does not always hold. Figure 1 shows the random walk segmentation result of a color image containing 4 regions (R_I, R_U, R_L, and R_O), starting from 4 seeds of 3 types (s_U, s_L, and s_O). The segmentation result is consistent with the image information, where all four regions of different color are separated. However, this is a counterexample to the connectedness proposition in [1] since the region R_I output by the segmentation does not contain any seed points. To see why this arises, we consider the image and seeds in more detail; readers should refer to Grady's original paper for an explanation of the algorithm. Let $D(R_1, R_2)$ be the color difference between regions R_1 and R_2. In this image, $D(R_I, R_U) = D(R_I, R_L) \gg D(R_U, R_L) \gg D(R_U, R_O) = D(R_L, R_O)$. Further, let U_O and L_O be the probability that a random walk starting at the center point of R_I first reaches seed type s_O via region R_U or R_L respectively. We experimentally observe that $U_U - U_O > U_O - U_L > 0$, which also matches with intuition. Thus $U_O + L_O = 2U_O > U_U + U_L = U_U + L_U = L_L + U_L$ (equality due to symmetry), i.e. the center point should be labeled the same as s_O. This gives an intuitive understanding of how isolated regions containing no seeds may occur, if the user fails to place an adequate number of seeds in suitable places.



(a) (b) (c) (d)

Figure 1: A counterexample: (a) input image containing 4 regions, and 4 seeds of 3 types; (b) random walk segmentation result; probability of a random walk starting at each pixel first reaching seed type s_U (c) or s_O (d).

3 Theoretical Analysis

We now consider where Grady's proof of connectedness breaks down. Following his notation, let x_i^s be the probability of a random walk starting at node i first arriving at a seed labelled s. In Proposition 1 of [1], although $\forall f \neq s, \exists i$ such that $x_i^s > x_i^f$, i can be different for each f. The proof of this proposition fails to show that any connected subset of unseeded nodes assigned to segmentation s must contain at least one seed labeled s. (Nevertheless, this proposition *does* hold if there are only two seed types, which is a special result of our Observation 1 in Section 3).

One may ask whether we can make some simple amendments, such as changing the assignment rule, to fix this problem. The answer is *'No'*. To show this, we analyze the random walk problem using an equivalent electrical circuit network. We further assume that the algorithm works on general graphs, and generalize the segmentation problem to a special convex hull partition problem (Observation 1). Then we show step by step (Proposition 1-5) that for any segmented graph which satisfies the connectedness property and has more than 3 label types, even allowing the algorithm to have more general weighting (Proposition 4) and assignment rule (Proposition 5), it is still possible to replace some part of it in a way which makes the connectedness property fail. Finally, Proposition 5 provides a condition under which random walk segmentation *is* guaranteed to give a connected result.

According to [2], a random walk graph has an equivalent electrical circuit network with conductances equal to the edge weights, and voltage sources replacing seed points. In this circuit network, the probability x_i^s is equal to the voltage at i if we give the seeds labeled s unit potential and other seeds zero potential. For a passive sub-network (PSN; without voltage sources) X of the circuit, we denote its boundary nodes as $N(X) = \{y_k\}$. Here boundary nodes are those outside X, having at least one neighbor within X; X comprises the nodes it contains, the conductance associated with these nodes, and the conductance between boundary nodes. Let the probabilities of a random walk starting at node p first reaching a seed of type $S = \{s_1, \cdots, s_m\}$ be a vector $x_p = [x_p^{s_1}, \cdots, x_p^{s_m}]$.

Observation 1. *For each node p in X, $x_p = \sum_k \lambda_p^k x_{y_k}$, $\lambda_p^k \geq 0$, $\sum_k \lambda_p^k = 1$, and the weights λ_p^k are uniquely determined by X. Let the vector be $\lambda_p = [\cdots, \lambda_p^k, \cdots]$.*

Using a circuit network analogy, each PSN X can be represented by a conductance matrix H (uniquely determined by X; see Proposition 1 for a detailed expression). The conductance matrix H for a circuit represents the linear relation between the inward current and boundary voltage, $I = HU$, where I and U are vectors formed by concatenating inward current and boundary voltage values at boundary nodes, and H is determined by the structure of the circuit network. If no output nodes are connected by zero resistance (infinite conductance), H is well-defined. To prove Proposition 3, we first analyze some properties (Proposition 1, 2) of the conductance matrix H:

Proposition 1. *Assuming there are M boundary points, $H = \{h_{ij}\}_{M\times M}$ has the following properties:*

1. *Symmetry: $H^T = H$.*
2. *Zero sum: $Hl_{M\times 1} = 0$, where the vector l has all elements 1.*
3. *Sign: $h_{ij} \geq 0$ for $i = j$ and $h_{ij} \leq 0$ for $i \neq j$.*

Proof. If C the $M \times M$ Laplacian matrix for boundary nodes, U_X the inner voltages, B the $n \times M$ negative connection matrix for connections between inner nodes and the boundary, and L is the $n \times n$ matrix for the inner nodes, Ohm's Law and Kirchhoff's Rules give

$$I = CU - \mathrm{diag}(B^T l)U + B^T U_X, \quad LU_X + BU = 0$$

Thus, $U_X = -L^{-1}BU$. It follows that $I = (-\mathrm{diag}(B^T l) - B^T L^{-1} B + C)U$. Hence:

1. $H = -\mathrm{diag}(B^T l) - B^T L^{-1} B + C = -\mathrm{diag}(B^T l)^T - (B^T L^{-1} B)^T + C^T = H^T$.
2. Using the fact that $Ll + B = 0, Cl = 0$, we have $Hl = -(\mathrm{diag}(B^T l) + B^T L^{-1} B)l = -B^T l - B^T L^{-1} B l = -B^T l + B^T l = 0$.
3. We have $B^T l \leq 0, L^{-1} \geq 0$, so $-\mathrm{diag}(B^T l) \geq 0, -B^T L^{-1} B \leq 0$. Thus $c_{ij} \geq 0$ for $i = j$ and $c_{ij} \leq 0$ for $i \neq j$, so property 3 follows from property 2.

$\square$

Proposition 2. *For any H satisfying Proposition 1, we can construct a PSN with conductance matrix H.*

Proof. The proof is simple—we just connect boundary nodes. For boundary nodes i and j, we add an edge with weight (conductance) $-h_{ij}$. It is easy to check that the conductance matrix is H. Note that no inner nodes are created, and such a matrix corresponds to C in the proof of Proposition 1. □

Proposition 3. *Given a conductance matrix $H = \{h_{ij}\}_{M\times M}$ and a set of vectors $\{\lambda_w\}$ satisfying $\lambda_w^k \geq 0$, $\sum_k \lambda_w^k = 1$, and $(h_{ij} = 0) \Rightarrow (\lambda_w^i \cdot \lambda_w^j = 0)$ for each $i < j$, we can construct a PSN X with conductance matrix H, containing nodes $\{w\}$ which has vectors $\{\lambda_w\}$.*

Proof. For each vector λ_w, we add an inner node w; for each boundary node i, we add an edge $e_{w,i}$ between w and i, with weight $\alpha_w \lambda_w^i$ (where α_w is a positive constant associated with w, to be determined later). Note that when $\lambda_w^i = 0$, the creation of $e_{w,i}$ is not needed. The conductance matrix associated with $e_{w,i}$ is $H_w = \{h_{w,ij}\}_{M\times M}$, where

$$h_{w,ij} = \begin{cases} -\alpha_w \lambda_w^i \lambda_w^j, & i \neq j \\ \sum_{k\neq i} -h_{w,ik}, & i = j \end{cases}.$$

Now, H_w is a valid conductance matrix, and $\lim_{\alpha_w \to 0} H_w = 0$. Thus, it is easy to check that small enough positive numbers $\{\alpha_w\}_n$ exist such that $H_S = H - \sum_w H_w$ is still a valid conductance matrix. After constructing a PSN for H_S as in Proposition 2, we get a PSN consisting of $n+1$ parts (n are created for vectors, and one is created by connecting to boundary nodes as in Proposition 2). These parts do not share nodes in the interior, so the overall conductance matrix is $\sum_w H_w + H_S = H$. Each node w has vector λ_w. □

From basic electrical circuit principles, if a PSN is replaced by another with the same conductance matrix, other parts of the circuit network are not affected. Proposition 3 shows that if X is a connected graph, for any set of voltages inside the convex hull of voltages of $N(X)$, we can design another sub-network X' to replace X, such that voltage and current at every node of the rest of the circuit does not change, and the nodes of X' give the set of specified voltages.

Proposition 4. *For a graph (circuit network), in which each node (pixel) i is associated with a 'color' c_i (scalar or vector), the weight for edge e_{ij} is given by $g(c_i - c_j)$, where the continuous function $g : V \to R^+$ satisfies $g(c) = g(-c)$, and $\lim_{\|c\|\to\infty} g(c) = 0$. Given a set of vectors as in Proposition 3, we can replace a PSN X by another PSN X', without changing other parts of the circuit network. The new network contains nodes with the given vectors and has the same weighting*

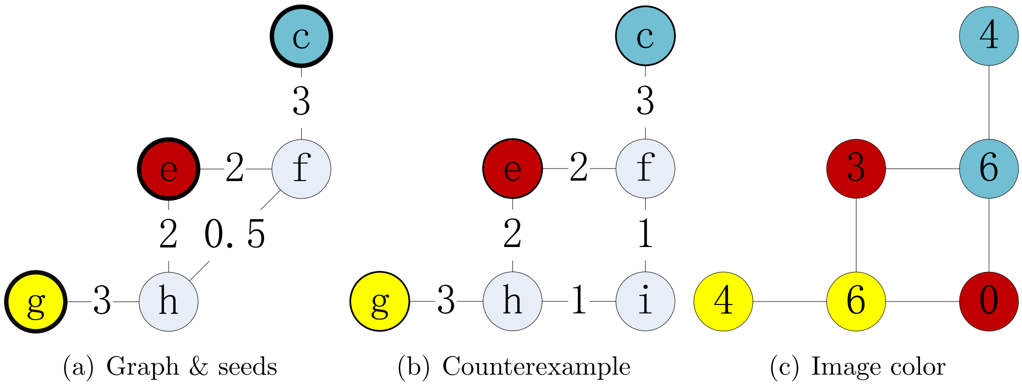


Figure 2: An example of counterexample construction: initial seeds are shown in color in (a) and the determined image colors and final pixel labels are shown in (c).

function (edge weights are uniquely determined by the 'color' of each edge's two end nodes).

Proof. We can replace X by X' to construct desired vectors while keeping the conductance matrix unchanged using a similar approach to that in Proposition 3. To construct 'colors' so that edges agree with the weighting function g, c_i for a newly created node i is firstly random initialized. For each edge $e_{ij} \neq g(c_i - c_j)$, let

$$w_{ij} = \frac{g(0)g(c_i - c_j)}{g(0) + g(c_i - c_j)}.$$

If $e_{ij} < w_{ij}$, we create another node j' with color c'_j, remove edge e_{ij}, and connect i, j' and j', j. The equivalent weight is $h(c'_j) = g(c_i - c'_j)g(c'_j - c_j)/(g(c_i - c'_j) + g(c'_j - c_j))$. Note that $h(c_i) = w_{ij} > e_{ij}$ and $\lim_{\|c'_j\| \to \infty} g(c'_j) = 0$, so $\exists c'_j \in V$ such that $h(c'_j) = e_{ij}$, i.e. the network remains unaltered. If $e_{ij} > w_{ij}$, let $k = \lfloor e_{ij}/w_{ij} \rfloor$, remove edge e_{ij} and create k nodes with color c_i, each connected to node i and j. If $e_{ij} - kw_{ij} > 0$, we create a node as above with weight $h(c'_j) = e_{ij} - kw_{ij}$. The total equivalent weight is just e_{ij}. □

Proposition 4 suggests that even in normal color based graph segmentation (the edge weights are determined by 'color' differences between nodes), counter examples can still be easily constructed by replacing a sub-network by another.

Figure 2(a) shows a small example which meets the connectedness property, with $x_f^{\{c,e,g\}} = \{0.55, 0.4, 0.05\}$ and $x_h^{\{c,e,g\}} = \{0.05, 0.4, 0.55\}$. While $x_i^{\{c,e,g\}} = \{0.3, 0.4, 0.3\}$ lies inside the convex hull of $\{x_f, x_h\}$, it does not belong to the same seed as either x_h or x_f. It can be constructed by replacing the PSN in (a) (only

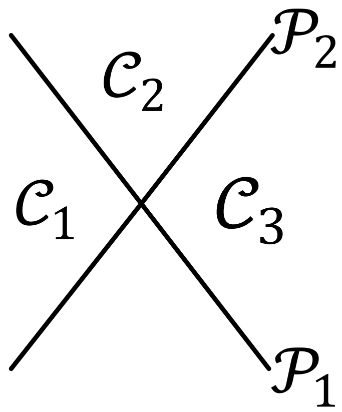


Figure 3: Convex sets separation example

containing the edge with weight 0.5) by another according to Proposition 3. Figure 2(c), constructed according to Proposition 4, is the image part corresponding to the counterexample in Figure 2(b).

From the above, we can design the interior voltages of a connected sub network provided that they lie inside the convex hull of voltages of the boundary.

Observation 2. *We define a segmentation to be a map f from the vector space $\{x_p\}$ to segmentation identifier—i.e., the segmentation is a partition of the convex hull generated by voltages at seed points. The segmentation method in [1] is thus a special case of segmentation, with x_{s_i} having i^{th} value 1, and other values 0, and $f(x_p) = \arg\max_i(x_p^i)$. Note that f is ill-defined when $\arg\max_i(x_p^i)$ has more than one value. Generally, such points have zero measure in the Euclidean space containing the convex hull generated by voltages at seed points; the convex hull has positive volume. Suppose the convex hull of seed voltages $\mathbb{C}$ is partitioned into $\{\mathbb{C}_i\}_{1\times m}$, with each $\mathbb{C}_i$ corresponding to a label type $x_{s_i} \in \mathbb{C}_i$, and having positive volume. For Grady's connectivity statement to be true, it requires that, for any $\{s_{i_k}\} \subset S$, $\bigcup \mathbb{C}_{i_k}$ is convex.*

Proposition 5. *The requirement in Observation 2 cannot be satisfied for $m \geq 3$.*

Proposition 5 shows that Grady's proof holds only for the two-label case, in which the line segment with end points x_{s_1}, x_{s_2} is partitioned into two segments at the midpoint $(x_{s_1} + x_{s_2})/2$.

Proof. Let the interior of $\mathbb{C}_i$ be $\mathcal{C}_i$ and the closure of $\mathcal{C}_i$ be $\bar{\mathcal{C}}_i$. Suppose $\{\mathcal{C}_i\}_{1\times m}$ satisfies the requirements in Observation 2 and $m \geq 3$. It is easy to show that $\mathcal{C}_i$ is convex and $\{\bar{\mathcal{C}}_i\}$ still satisfies the requirements in Observation 2. Then, according to the separation theorem of convex sets, there exist hyperplanes $\mathcal{P}_1, \mathcal{P}_2$, separating

$\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ as shown in in Figure 3, and $\mathcal{P}_i \cup \mathcal{C}_j = \emptyset, i = 1, 2, j = 1, 2, 3$. We choose $P_i \in \mathcal{C}_i, i = 1, 2, 3$, and $K_1 = P_1P_2 \cap \mathcal{P}_1, K_2 = P_2P_3 \cap \mathcal{P}_2$, where P_iP_j denotes the line segment with end points P_i and P_j. We choose a sequence, $W_1, W_2, \cdots, W_n \cdots$, satisfying $W_n \in P_1K_1, W_n \neq K_1$, and $W_n \to K_1$. Note that $\bar{\mathcal{C}_1} \cup \bar{\mathcal{C}_2}$ is convex, and $W_n \in \bar{\mathcal{C}_1} \cup \bar{\mathcal{C}_2}$. As $W_n \notin \bar{\mathcal{C}_2}$, we have $W_n \in \bar{\mathcal{C}_1}$. The closedness of $\bar{\mathcal{C}_1}$ implies that $K_1 \in \bar{\mathcal{C}_1}$. Similarly, $K_2 \in \bar{\mathcal{C}_3}$. But obviously, the interior of segment K_1K_2, $(\lambda K_1 + (1 - \lambda)K_2) \notin \bar{\mathcal{C}_1} \cup \bar{\mathcal{C}_3}$, $0 < \lambda < 1$. This contradicts the requirement of convexity of $\bar{\mathcal{C}_1} \cup \bar{\mathcal{C}_3}$. □

4 Conclusion

We have given a counterexample to disprove Grady's assertion concerning the connectedness of segmentations produced by random walk [1]. Further theoretical discussions show that the original assertion is not true in its most general form. Despite this deficit, experiments on many real world images do result in connected segmentations—random walk segmentation is indeed a powerful tool in many situations. Our discussion gives a new way to understand the structure of random walk segmentations, and what users can expect from random walk segmentation.

Acknowledgements

We thank the reviewers for their helpful suggestions and Professor Ralph Martin for his valuable comments. This work was supported by the National Basic Research Project of China (Project Number 2006CB303106), the National High Technology Research, the Development Program of China (Project Number 2009AA01Z330) and the Intel-MoE Joint Research Fund (MOE-INTEL-08-01).

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